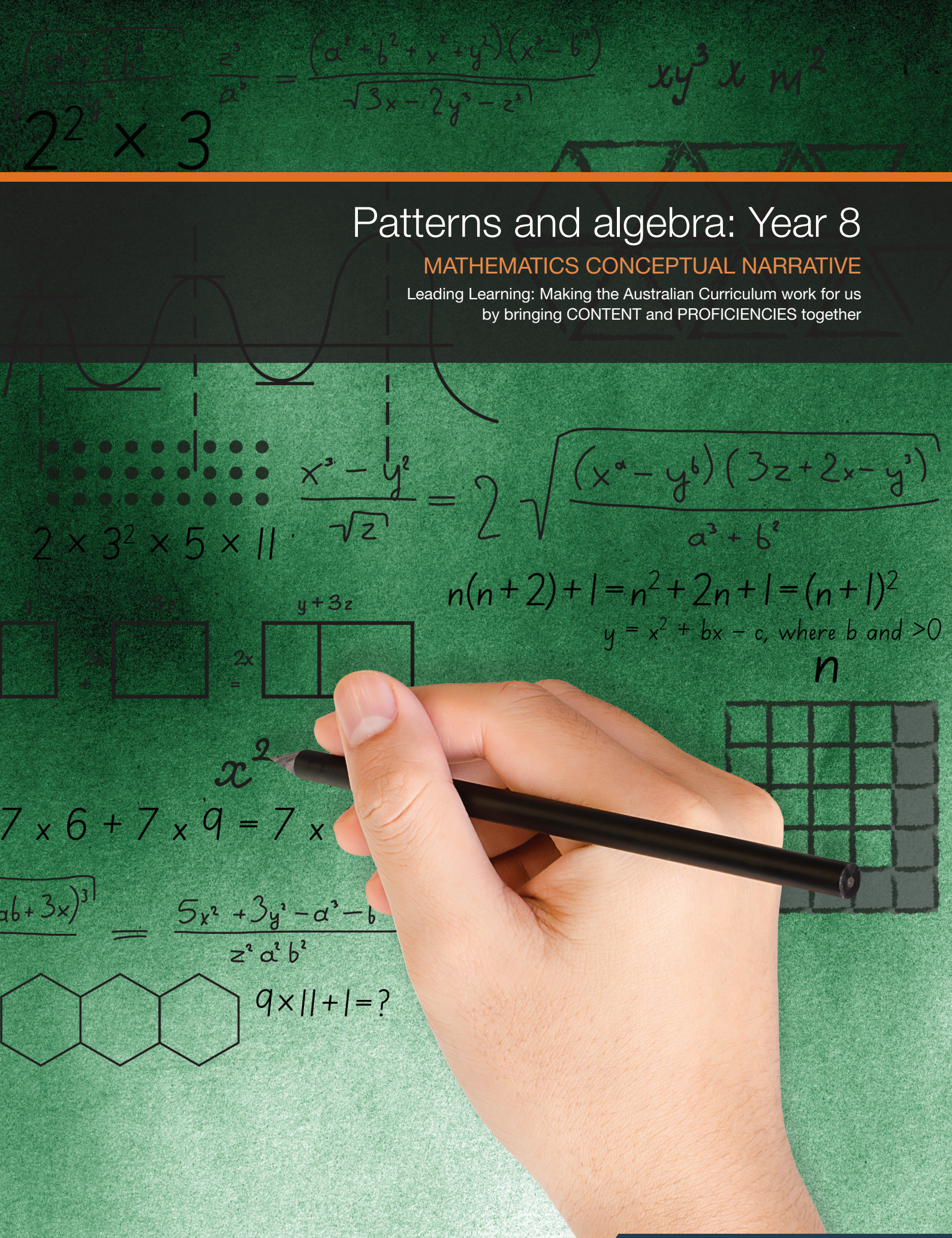
Patterns and algebra: Year 8

MATHEMATICS CONCEPTUAL NARRATIVE

Leading Learning: Making the Australian Curriculum work for us
by bringing CONTENT and PROFICIENCIES together



www.acleadersresource.sa.edu.au

Government of South Australia
Department for Education

Contents

Resource key

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

More information about 'Transforming Tasks':
http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

Bringing it to Life (BitL) key questions are in bold orange text.

Sub-questions from the BitL tool are in green medium italics – these questions are for teachers to use directly with students.

More information about the 'Bringing it to Life' tool:
http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

Throughout this narrative—and summarised in **'Patterns and algebra' from Foundation to Year 10A** (see page 22)—we have colour coded the AC: Mathematics year level content descriptions to highlight the following curriculum aspects of working with patterns and algebra:

- Copy, continue and create patterns
- Investigate and describe number patterns
- Use variables to represent numbers and create algebraic expressions
- Simplify and identify equivalent algebraic expressions by extending and applying laws and properties of numbers
- Use algebraic thinking and processes to solve problems.

What the Australian Curriculum says about ‘Patterns and algebra’

Content descriptions

Strand | Number and algebra.

Sub-strand | Patterns and algebra.

Year 8 ◆ | ACMNA190
Students extend and apply the distributive law to the expansion of algebraic expressions.

Year 8 ◆ | ACMNA191
Students factorise algebraic expressions by identifying numerical factors.

Year 8 ◆ | ACMNA192
Students simplify algebraic expressions involving the four operations.

Year level descriptions

Year 8 ◆ ◆ | Students describe patterns involving indices and recurring decimals, and identify commonalities between operations with algebra and arithmetic.

Achievement standards

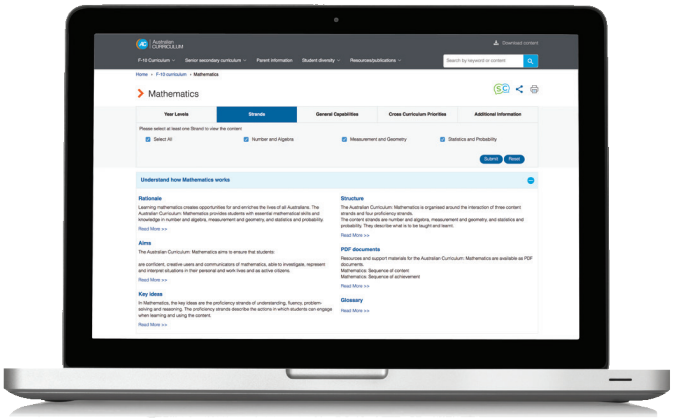
Year 8 ◆ | Students make connections between expanding and factorising algebraic expressions.

Year 8 ◆ | Students simplify a variety of algebraic expressions.

Numeracy continuum

Recognising and using patterns and relationships

End of Year 8 ◆ | Students identify trends using number rules and relationships (Recognise and use patterns and relationships).



Source: ACARA, Australian Curriculum: Mathematics

Working with Patterns and algebra

Important things to notice about this sub-strand of the Australian Curriculum: Mathematics and numeracy continuum

What we are building on and leading towards in Year 8 'Patterns and algebra'

In Year 7 the concept of a variable as a way of representing numbers using letters is introduced. Students still create and evaluate expressions, just as they had in previous years; but now they will create algebraic expressions instead of number sentences. This transition from writing number sentences to algebraic expressions requires the students to apply the laws of arithmetic to algebra. This application begins in Year 7 when students will apply the associative [a+(b+c) = (a+b)+c] and commutative laws [a+b = b+a] to algebraic expressions in order to simplify them.

In Year 8 students begin to expand algebraic expressions applying the distributive law [a x (b+c) = a x b + a x c] to algebra. They simplify more complex expressions that may include brackets and any or all of the four operations. The relationship between expansion and factorisation is also explored as students begin to factorise using a numerical factor.

In Year 9 the application of the laws of arithmetic to algebra continues and now includes the index laws; but only using positive or zero integers at this stage. Students continue to apply the distributive law, but to more complex expressions, including binomials.

In Year 10 students continue to simplify, factorise and expand algebraic expressions. Factorisation at this level will now include taking out an algebraic factor of monic algebraic expressions and indices could include positive or negative integers. Simplification now includes algebraic fractions with all four operations.

In Year 10A students apply long division to polynomials and they investigate the relationship between this, and the factor and remainder theorems.

- **Notice that** algebraic thinking can depend on the ability to recognise patterns as well as a highly developed number sense; in particular, fluency with number facts and multiplicative rather than additive thinking.
- It is common for teachers and students to overvalue algebraic expressions and formulae at this stage of development. Students can have a deep understanding of the patterns and relationships they observe and be able to extend and generalise their observations, and even be able to explain and write about it; yet have no concept of how this relates to the abstract representation with pronumerals. Hence the statement, '*I understood maths until they introduced the alphabet*'.
- Similarly, students can be very proficient in simplifying algebraic expressions, substitution, rearrangement and solving equations; but have no conceptual understanding of how this relates to the properties of number, or the power and purpose of algebra, '*There must be a formula for this*'.
- As with all branches of mathematics, it is important to develop concepts from conceptual understanding. Encourage learners to explain the sense they have made of it in their own words, then make the connections with the formal mathematical language and symbols, using both until the learners adopt the mathematical language as their own.

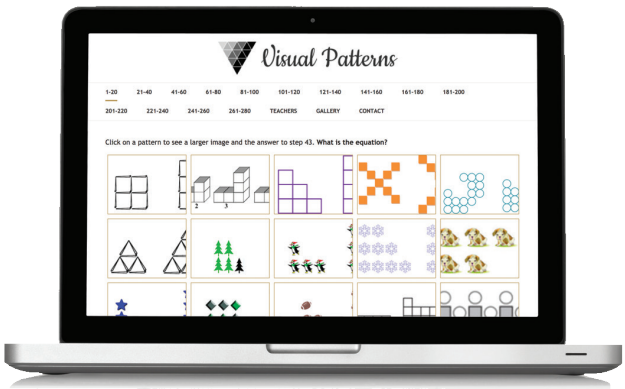
Engaging learners

Classroom techniques for teaching Patterns and algebra

Visual patterns

This website has multiple visual patterns that students can consider and describe in a way that makes sense to them. Students will have multiple interpretations of each image, promoting multiple ways of visualising, solving problems and stimulating dialogue. There is no right, or wrong answer and all learners can participate while you stay in the 'opinion space'. They can be used as lesson starters.

Visual patterns can be found at:
http://www.visualpatterns.org/



Source: *Visual Patterns*, visualpatterns.org, 2017

Counter-intuitive experiences

Counter-intuitive experiences intrigue students who want to make sense of what they have seen. This is a way to cognitively engage students to explore the phenomena more closely and use or be convinced, by the mathematics that explains it.

The Flash Mind Reader is an impressive 'confidence trick'.

An electronic version of the game can be found at:
http://www.flashlightcreative.net/swf/mindreader/

Source: *The Original Flash Mind Reader*, flashlight creative

From tell to ask

Transforming tasks by modelling the construction of knowledge (Examples 1–5)

The idea that education must be about more than transmission of information that is appropriately recalled and applied, is no longer a matter for discussion. We know that in order to engage our students and to support them to develop the skills required for success in their life and work, we can no longer rely on a 'stand and deliver' model of education. It has long been accepted that education through transmission of information has not worked for many of our students. Having said this, our classrooms do not necessarily need to change beyond recognition. One simple, but highly effective strategy for innovation in our classrooms involves asking ourselves the question:

What information do I need to tell my students and what could I challenge and support them to develop an understanding of for themselves?

For example, no amount of reasoning will lead my students to **create the names of the distributive and commutative laws** themselves. They need to receive this information in some way. However, it *is* possible my students can be challenged with questions that will result in them **identifying generalised patterns from their number facts**, so I don't need to instruct that information.

At this stage of development, students can **develop an understanding of equivalent algebraic expressions by generalising from their number facts**. When teachers provide opportunities for students to **identify and describe the number properties used to simplify calculations**, they require their students to generalise from the classifications using algebraic thinking. Telling students algebraic rules removes this natural opportunity for students to make conjectures and verify and apply connections that they notice. Using questions such as the ones described here, supports teachers to replace 'telling' the students information, with getting students to notice for themselves.

When we are feeling 'time poor' it's tempting to believe that it will be quicker to tell our students a formula, rather than ask a question (or series of questions) and support them to establish a formula for themselves. Whether this is true or not really depends on what we have established as our goal. If our goal is to have students recall and apply a particular rule or theorem during the current unit of work, then it probably is quicker to tell them the rule and demonstrate how to apply it. However, when our goal extends to wanting students to develop conceptual understanding, to learn to think mathematically, to have a self-concept as a confident and competent *creator and user* of mathematics, then telling students the rule is a false economy of time.

When we challenge our students to **establish a theorem**, we model that algebra can be powerful and useful. We provide our students with an authentic context for working algebraically. Telling students formulae removes this opportunity for students to generalise.

Teachers can support students to understand the factorisation of numbers by asking questions as described in the **Understanding** proficiency: **What patterns/connections/relationships can you see?** The intent of this question is to promote learning design that intentionally plans for students to develop a disposition towards looking for patterns, connections and relationships.

Curriculum and pedagogy links

The following icons are used in each example:

The '**AC**' icon indicates the Australian Curriculum: Mathematics content description(s) addressed in that example.

The '**Bringing it to Life (BitL)**' tool icon indicates the use of questions from the *Leading Learning: Making the Australian Curriculum Work for Us* resource.

The Bringing it to Life tool is a questioning tool that supports teachers to enact the AC: Mathematics Proficiencies: http://www.acleadersresource.sa.edu.au/index.php?page=bringing_it_to_life

The '**From tell to ask**' icon indicates a statement that explains the transformation that is intended by using the task in that example.

This idea of moving 'From tell to ask' is further elaborated (for Mathematics and other Australian Curriculum learning areas) in the 'Transforming Tasks' module on the *Leading Learning: Making the Australian Curriculum work for Us* resource: http://www.acleadersresource.sa.edu.au/index.php?page=into_the_classroom

Look out for the purple pedagogy boxes, that link back to the SA TfEL Framework.

From tell to ask examples	
Example 1: Hexagonal train – identifying equivalent algebraic expressions Students simplify algebraic expressions involving the four operations.	ACMNA192 ◆
Example 2: Distributive law – number facts Students extend and apply the distributive law to the expansion of algebraic expressions.	ACMNA190 ◆
Example 3: Distributive law – area model Students extend and apply the distributive law to the expansion of algebraic expressions.	ACMNA190 ◆
Example 4: Flash mind reader – simplifying algebraic expressions Students simplify algebraic expressions involving the four operations.	ACMNA192 ◆
Example 5: Factorising with tiles – area model Students factorise algebraic expressions by identifying numerical factors.	ACMNA191 ◆

Example 1: Hexagonal train – identifying equivalent algebraic expressions

ACMNA192 ◆
Students simplify algebraic expressions involving the four operations.

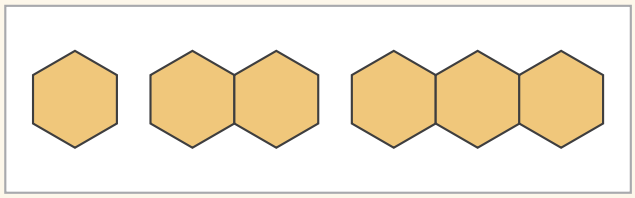
Questions from the BitL tool

Understanding proficiency:
What patterns/connections/relationships can you see?
Can you represent/calculate in different ways?

Reasoning proficiency:
In what ways can your thinking be generalised?
What can you infer?

Instead of ***telling*** students about equivalent algebraic expressions, we can challenge students to recognise the equivalence for themselves, by ***asking*** questions.

Begin a discussion by explaining that a series of trains of different lengths, can be made out of hexagonal shapes.



Ask students:

- *What quantities do you notice varying from one train to the next?* (The number of hexagons, the area of the train, the perimeter, the number of edges, the number of vertical edges, the number of corners, the number of joins … etc.)

The two variables I want to know more about are the number of hexagons and the perimeter of the shape (train).

- *What do you think the perimeter of the train made from 10 hexagons might be?*
- *How might you find out?*
- *Convince me. Can you do it another way? Could you use a table? A graph?*

There are many levels of entry for this task. Students can:

- Build the train and count. (It is a common misconception that the perimeter of the second train is 12, which you can encourage students to self-correct by asking them to show you how they determined that. Ask students that have the correct answer as well.)
- Draw a table using values from the first 3 and then continue the pattern for each train by adding on 4 each time, up to the 10th term.

Term	P
1	6
2	10
3	14

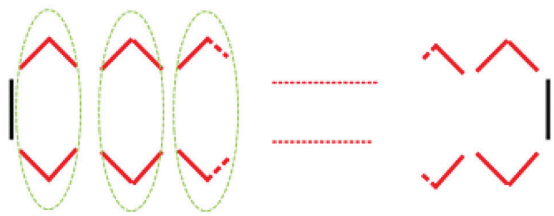
+4
+4

- Begin the table then realise that it would be 6, with 9 lots of 4 added on. (It is a common misconception that it would be 10 lots of 4. Ask them what they would have thought the 3rd train would have been with that thinking (6 + 3x4), yet it is actually 14, which is 6 + 2x4.)

  t = 1 P = 6 + 0x4
  t = 2 P = 6 + 1x4
  t = 3 P = 6 + 2x4
  …
  t = 10 P = 6 + 9x4

- Rather than do an iterative pattern, students might identify a relationship between the two variables. (The perimeter is 4 times the term, plus 2 for each train (4t+2). Check that this works for the trains you have (4x1 + 2 = 6, 4x2 + 2 = 10, etc.))

While this pattern works for the values in the table it may not work for all trains, we need to think **why** this works. (See Figure 1.)

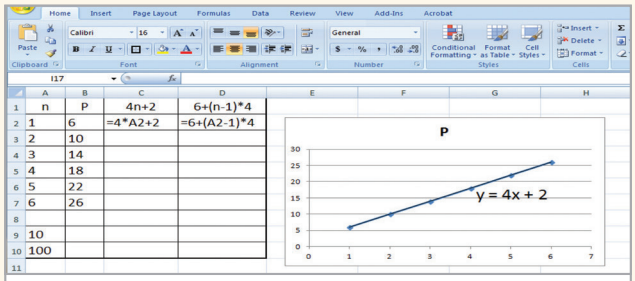


There is one side at the start and one at the end, making 2. Then there is a set of 4 for each hexagon in the train (4t+2).

Figure 1

This is an opportunity to consider how all these expressions can be explained by considering the geometrical features of the train, and yet they look so different. How could we verify that the expressions were all equivalent?

Using an Excel spreadsheet, students can use two or more of the expressions as formulae to verify that they have the same values for a large number of train lengths.



Fill down, draw a chart, fit a trendline.

Ask students:

- *How many examples do you need to check to prove that they are the same?*
- *What would be needed to show they were not the same expressions?* (No number of examples would be enough to **prove** that they were the same for all possible numbers, but just one example that does not work is enough to prove they are different.)

Construction of knowledge: Building a bridge between their reasoning and formal mathematical language

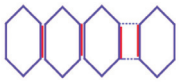
When students have identified a pattern and/or have been able to generalise:

- ask them to explain their thinking verbally and in writing (even writing four and plus instead of 4 and +)
- scaffold how this could be recorded using mathematical symbols in a precise and accurate way.

This is an example of how to scaffold the connection between the students' reasoning about the hexagonal train perimeter and a more mathematical way of recording their pattern. While this is an important connection to be made, it must not be considered more important than conceptual understanding and being able to explain the concept in a way that makes sense to them. To support this transition, ask students to relate what they are doing back to the exact problem they did when exploring, 'How is this the same/different to what you did when …'.

Encourage and reward the use of mathematical symbols but require all students to justify their thinking, even if they need to write this in a worded response. This is time consuming and, in most cases, is sufficient motivation for them to adopt at least some of the mathematical language which simplifies their recording.

There are multiple ways to describe the pattern:

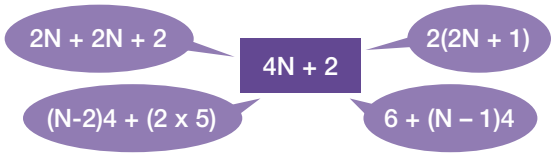
	
6 + (n-1) x 4	**6n – (n-1) x 2**
Begin with the six (**6**) sides, pull one of the sides out and add (**+**) four (**4**) sides, (2 at the top and 2 at the bottom) for every (**x**) hexagon except the first one (**n-1**) in the train.	A number of hexagons (**n**) each have 6 sides (**x6**) but you lose (**-**) 2 sides when you (**2x**) join them together. There are always one less joins than hexagons (**n-1**).

The coloured annotations show how the algebraic connections can be made with the text.

A convincing argument can be made to support the pattern by relating it to building the train. This inductive reasoning is a mathematical proof because it is a valid convincing argument.

Identifying a pattern for a few values in a table does not constitute a proof, as there is no reason why the pattern must continue or why it would be true for every case. It is only a conjecture (hypothesis). If you check a few more cases, you are verifying the rule but not proving it.)

Can you work out how the students might have come up with these expressions?



- *Can you generalise? Can you write a description for a Year 7 student to explain how you would determine the perimeter of a train of 100 hexagons? What about any length?*
- *Is it possible to have a train with a perimeter of 2016? Convince me.*

In fact, the explanations that link the expressions to the geometrical features of the train are proof that **the algebraic expressions are the same** for all positive integers, as they are all correct answers to the same problem.

A more formal proof that the expressions are the same, can be found using the laws of algebra which are really the properties that work for all numbers. For example, 2 lots of any number plus another 2 lots of the same number, is 4 lots of that number (2N +2N = 4N).

Many of these equivalent expressions can be demonstrated using manipulatives.

Consider: 4 x (n – 1) + 6 (see Figure 2).

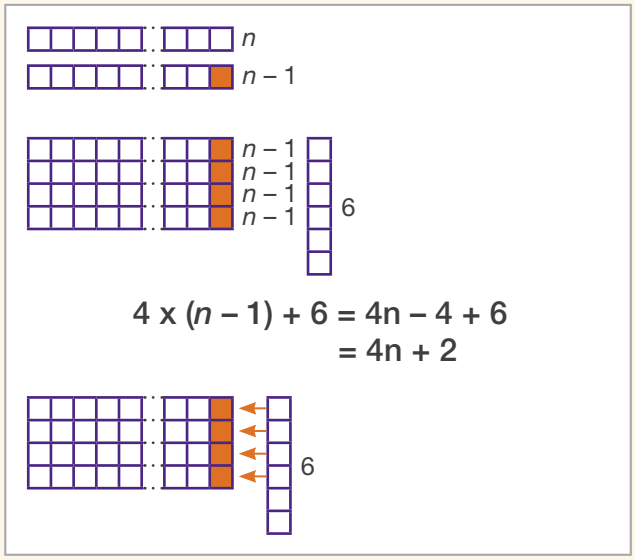


Figure 2

(A similar activity also appears in the *Linear and non-linear relationships: Year 8* narrative.)

Example 2: Distributive law – number facts

ACMNA190 ◆
Students extend and apply the distributive law to the expansion of algebraic expressions.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about the distributive law, we can challenge students to transfer their knowledge about number facts to recognise the generalisation for themselves, by ***asking*** questions.

Begin a discussion by considering the following products:

101 x 17 **1002 x 81** **43 x 31** **22 x 35**

Ask students:

- *Evaluate the products in as many ways as possible.* (Share the different approaches, such as a calculator, the long multiplication algorithm, and considering factors (22 x 35 = 11 x 2 x 7 x 5 = 77 x 10), etc.)

Identify that the approach you want to focus on is considering 101 x 17 as (100 x 17 + 1 x 17):

- *Could you find other products where this technique might be helpful?* (It is most useful when one of the numbers is 1 or 2 more than a multiple of 10. Students might also suggest when one of the numbers is 1 or 2 less than a multiple of 10, eg 99 x 37. You can extend students to consider applying the technique in two different ways for a product like 101 x 1002.)
- *Could you find products where it might not be helpful? Why not? Would it always give the correct answer even in these cases?* (Using products such as 37 x 54 apply the technique in two different ways, for example, ((30 + 7) x 54) and (37 x (50 + 4)). Students can determine that neither greatly simplifies the calculation, but both give the correct answer.)
- *How might you explain this technique in words so that other Year 8s could apply it to any product? In symbols?* (It is quite difficult to explain this technique in general terms, but it is important that they can express their understanding of the pattern. After they can write this, match the algebraic representation to the text. The brevity of the symbolic expression supports the use of algebra to simplify description.

  If you want a certain number of lots (like 101) of another number, you can also work it out by finding two numbers that add up to the first number (like 100 and 1) and then, for each of the two numbers, find that many lots of the second number (100 lots and 1 lot) and add them together.)

(a + b)	x	c
a x c	+	b x c

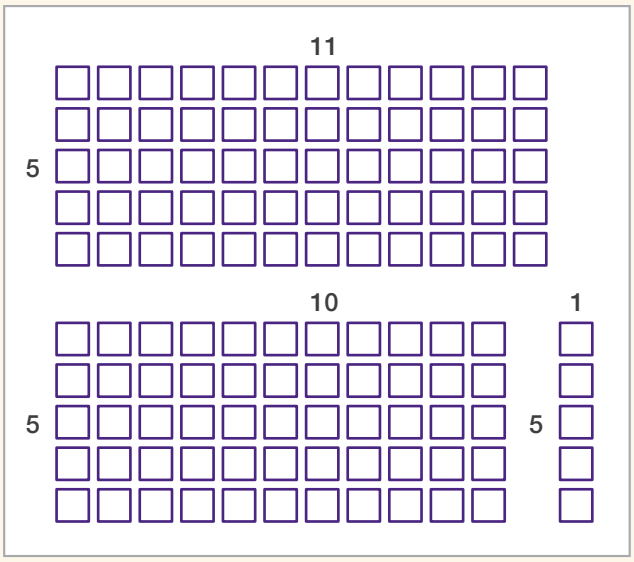
- *How might you match your general description to other products where you used this technique? What would a, b and c be?*
- *Consider a product like 47 x 1001. How is this the same or different to the generalisation (a + b) x c?* ((47 x 1001 = 1001 x 47) and so can be done like the others, (a + b) x c, but it is also true that 47 x 1001 is 47 lots of 1000 plus 47 lots of 1 (47 x 1001 = 47 x 1000 + 47 x 1), using the rule (a x (b + c)) = (a x b + a x c).)

Manipulatives can be used to demonstrate these properties of numbers:

Consider 5 x 11.

Set up the tiles to show 5 x 11.

Show by grouping the counters, that this number can be written as 5 lots of 10, plus 5 lots of 1.



Show students how this could be recorded using a numerical expression:

5 x 11 = 5 x (10 + 1)
= 5 x 10 + 5 x 1
= 50 + 5
= 55

Note how the equal signs are under each other and we read down the page.

Consider 11 x 5.

Set up the tiles to show 11 x 5.

Ask students:

- *How is this pattern the same/different to the pattern of tiles you set up for 5 x 11?* (The same arrangement rotated through a quarter turn can illustrate this number fact.)

Show by grouping the counters, that this number can be written as 10 lots of 5, plus 1 lot of 5.

Construction of knowledge: Building a bridge between their reasoning and formal mathematical language

When students have applied their conceptual understanding to solve a problem:

- ask them to explain their thinking verbally
- scaffold how this could be recorded using mathematical symbols in a precise and accurate way.

This is an example of how to scaffold the connection between the students' reasoning and a more mathematical way of recording their thinking. While this is an important connection to be made, it must not be considered more important than conceptual understanding and being able to explain the concept in a way that makes sense to them. Encourage and reward the use of mathematical symbols but require all students to justify their thinking even if they need to write this in a worded response. This is time consuming and, in most cases, is sufficient motivation for them to adopt at least some of the mathematical language which simplifies their recording.

Example 3: Distributive law – area model

ACMNA190 ◆
Students extend and apply the distributive law to the expansion of algebraic expressions.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about the distributive law, we can challenge students to transfer their knowledge about area to recognise the generalisation for themselves, by ***asking*** questions.

Start this activity by instructing students to calculate 23 x 17 in as many different ways as possible and share their methods.

Discuss the following:

'Another Year 8 student did it this way. Can you explain their thinking?'

	17
20	340
3	51

340 + 51 = 391

- *Is there more than one way to use this method to find 23 x 17?* (It may be done as 23 by (20 + 3).)
- *How might you use this method to calculate other products? How might you check?*

The student thought that it could also be done using four areas instead of two.

- *How might you use four rectangles to find 23 x 17? How might you check?*
- *Can you generalise?*
- *Why does this work?*
  (It is a repeated use of the distributive law.)

23 x 17
= (20 + 3) x 17
= 20 x 17 + 3 x 17
= 20 x (10 + 7) + 3 x (10 + 7)
= 20 x 10 + 20 x 7 + 3 x 10 + 3 x 7
= 200 + 140 + 30 + 21

	10	7
20	200	140
3	30	21

200 + 140 + 30 + 21 = 391

- *Could you find products where it might not be helpful? Why not? Would it always give the correct answer even in these cases?*
- *How might you explain this technique in words so that other Year 8s could apply it to any product? In symbols?* (It is quite difficult to explain this technique in general terms, but it is important that they can express their understanding of the pattern. After they can write this, match the algebraic representation to the text. The brevity of the symbolic expression supports the use of algebra to simplify description.)

  If you want a certain number of lots (like 101) of another number (like 52), you can also work it out by finding two numbers that add up to the first number (like 100 and 1) and also two numbers that add up to the second number (like 50 and 2). The product can be found by considering the sum area of the four parts of the rectangle, where each of the first two numbers gets multiplied by each of the second two eg (100 x 50 + 100 x 2 + 1 x 50 + 1 x 2).

	c	d
a	ac	ad
b	bc	bd

(a + b)(c + d) = ac + ad + bc + bd

a x c	+	a x d	+	b x c	+	b x d

Example 4: Flash mind reader – simplifying algebraic expressions

ACMNA192 ◆
Students simplify algebraic expressions involving the four operations.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/relationships can you see?

Can you represent/calculate in different ways?

Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about the use of algebra, we can challenge students to transfer their knowledge about generalisations for themselves, by ***asking*** questions.

The Flash Mind Reader is an impressive 'confidence trick'. An electronic version of the game can be found at this website: http://www.flashlightcreative.net/swf/mindreader/

Select any 2-digit number. Add the two digits together and subtract the total from the original number, eg 23:
2+3 = 5
23–5 = 18.

Repeating this several times reveals that you always seem to get a multiple of 9. This is the basis of the mindreading trick. In the table of 2-digit numbers, the symbol next to all multiples of 9 is the same and so the 'mind reader' will always select the correct symbol.

Discuss with students:

- *Why does it work?*
- *Do you notice any patterns in our answers?*
- *Would this always work? How could we check? How many numbers would we need to check to be sure?* (There are 90 2-digit numbers that would need to be checked.)
- *How might we use algebra to help us understand what is happening?*

For a 2-digit number, **ab**, the value of the number is (10 x a + b) = (**10a + b**).

If the two digits are added together, the value is **a + b**.

If this result is subtracted from the original number:

$$\begin{array}{r} 10a + b \\ -\quad a + b \\ \hline 9a \end{array}$$

Check using the distributive law (10a+b) – (a+b)

Hence the answer is always a multiple of 9.

Interesting questions to ask students:

- *What do you notice about the symbol next to the multiples of 9 in the table next to the Mind Reader?* (The symbols are all the same for each multiple of 9, but change each turn.)
- *How does the multiple of 9 that is the answer, relate to the original number?* (If the final answer is 36, then 9**a** = 36 and **a** = 4. a is the tens digit of the original number, so I know it was a number in the forties.)
- *Why is it that you can determine the tens digit of the original number, but not the units?* (The units digit is eliminated when we subtract.)
- *Why does the table have five columns of 20 numbers from 0 to 99 and is not set up like a 100 frame (10 rows of 10 numbers), like you may expect with a normal grid of numbers?* (If the numbers were in a 100 frame, the multiples of 9 would line up diagonally; therefore the pattern of same symbols would be obvious.)
- *Would this work with 3-digit numbers? How might you adapt it so that you could identify the symbol regardless of the starting number?*

Find other number tricks or make up some of your own:

Choose a number.	n
Add 5.	$n + 5$
Double the result.	$2n + 10$
Subtract 4.	$2n + 6$
Divide the result by 2.	$n + 3$
Subtract the number you started with. **The result is 3. Why?**	3

Check using the distributive law, simplify $\frac{(2(n+5)-4)}{2} - n$

Example 5: Factorising with tiles – area model

ACMNA191 ◆
Students factorise algebraic expressions by identifying numerical factors.

Questions from the BitL tool

Understanding proficiency:

What patterns/connections/ relationships can you see?

Can you represent/calculate in different ways?

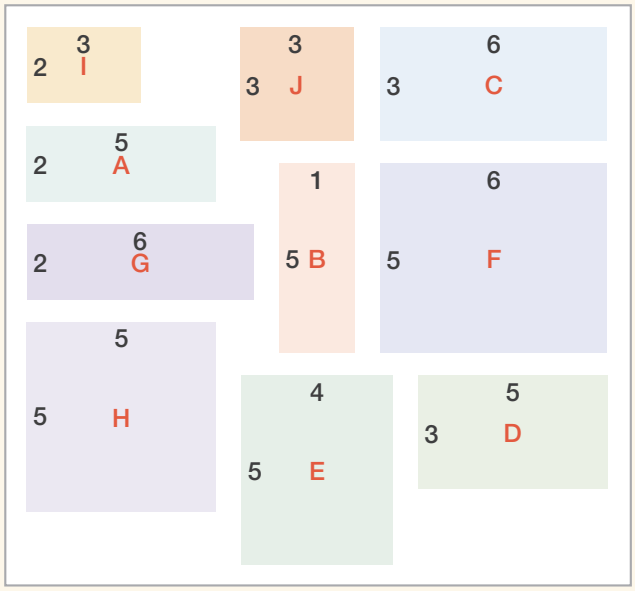
Reasoning proficiency:

In what ways can your thinking be generalised?

What can you infer?

Instead of ***telling*** students about the factorising, we can challenge students to transfer their knowledge about area to recognise the generalisation for themselves, by ***asking*** questions.

Begin this activity by giving students a pack of coloured rectangles, of which groups of 2, 3 and/or 4 have the same length but a different width. Ask students to use two of them in different ways to make bigger rectangles.

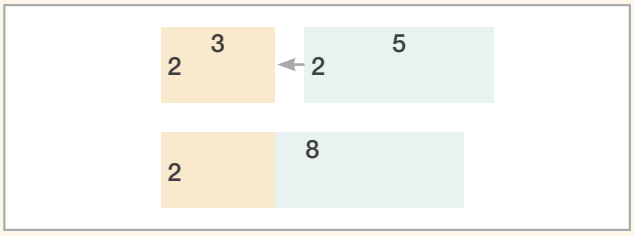


Ask students:

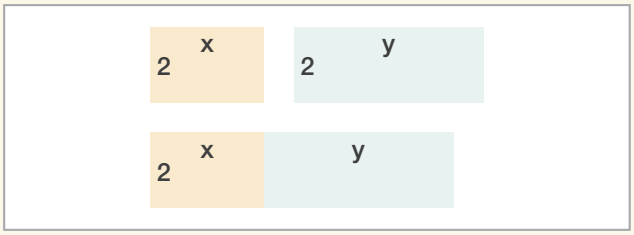
- *Which rectangles did you put together? What conditions are necessary for the rectangles to form a single larger rectangle?*
- *How could you record the combinations you have made?*

A student who joined **Rectangle I** and **Rectangle A**, recorded it using the number sentence

2x3 + 2x5 = 2 x (3+5).



- *Does that make sense to you? How might you record your combinations using number sentences?*
- *How might you generalise for rectangles with a width of 2? How could that be written as an algebraic sentence?*



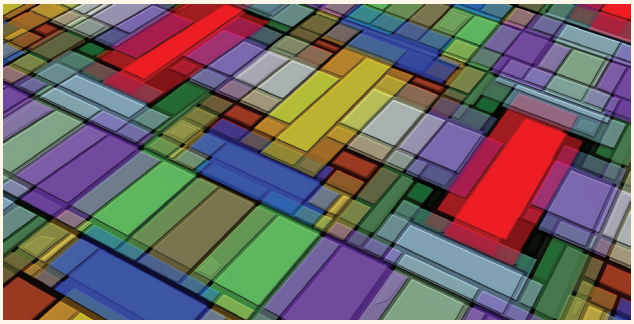
Challenge your students by asking the following:

- *Can more than two be put together to make a single rectangle? How might you record that? Could you generalise?*
- *Could all the cards be put together to make one large rectangle?*
- *How might you illustrate the following number sentence 4ab + 6a = 2a (2b +3)?*

The following is a possible class extension activity.

Will they all fit together?

Students can design the internal and overall dimensions of a tile mosaic made of individual units, to create an extreme number sentence or sentences. This could be planned in conjunction with another learning area.



Source: wobogre, Pixabay

Proficiency: Problem-solving

Proficiency emphasis and what questions to ask to activate it in your students (Examples 6–9)

Problems are described in the AC: Mathematics as 'meaningful' or 'unfamiliar'. Students of *all* abilities and ages should be provided with experiences of both meaningful and unfamiliar problems.

Meaningful problems

Meaningful problems are set in a context that a student can project themselves into. It may be that the mathematics and strategy being applied is familiar to the student or the problem relates to their own life experience. Connecting with a context does not mean that the students have to see it as 'fun' nor does it have to relate to an immediately practical situation from daily life.

Students can connect with a problem through provocations such as the use of manipulatives (either physical or digital) or through a dramatisation (eg, a story, interesting background information, a video clip). The intention is to give students the opportunity to work as a mathematician would work, in a context that they can access at their current stage of development.

Unfamiliar problems

Fundamentally there are two groups of unfamiliar problems:

- Problems for which the students would not be able to say that they had done a similar example previously, they would therefore need to create an approach (develop a strategy).
- Problems in which the students develop a new piece of knowledge. They begin the problem by applying the knowledge/skills that they have and they complete the problem having recombined that knowledge to form a new piece of understanding.

Growth mindset: Learning that not knowing is the beginning of a learning opportunity

Unfamiliar problems tend to provoke a response of, 'I don't know', or 'I'm not sure'. Students respond differently to this feeling; some shut down, others begin to ask, 'But how could I work that out?'

In developing powerful learners we are aiming for all of our students to learn that 'not knowing' is the *beginning of a learning opportunity* and that the first move that they need to make on the journey to finding out more is to ask, 'What could I do to work this out?'

Engaging in problem-solving supports the move *from tell to ask*

Instead of ***telling*** students:

- the problem to solve
- the information they'll need
- the steps they should take.

We can ***ask students to identify***:

- the problem to solve
- the information they'll need
- a possible process to use.

Proficiency: Problem-solving examples	
Example 6: More number pyramids – finding patterns Students simplify algebraic expressions involving the four operations.	**ACMNA192** ◆
Example 7: Designing table mats and mosaics – distributive law Students extend and apply the distributive law to the expansion of algebraic expressions.	**ACMNA190** ◆
Example 8: Stacking dice – finding a pattern Students extend and apply the distributive law to the expansion of algebraic expressions.	**ACMNA190** ◆
Example 9: Cracking the concrete – finding a pattern Students simplify algebraic expressions involving the four operations.	**ACMNA192** ◆

Example 6: More number pyramids – finding patterns

ACMNA192 ◆
Students simplify algebraic expressions involving the four operations.

Questions from the BitL tool

Problem-solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.

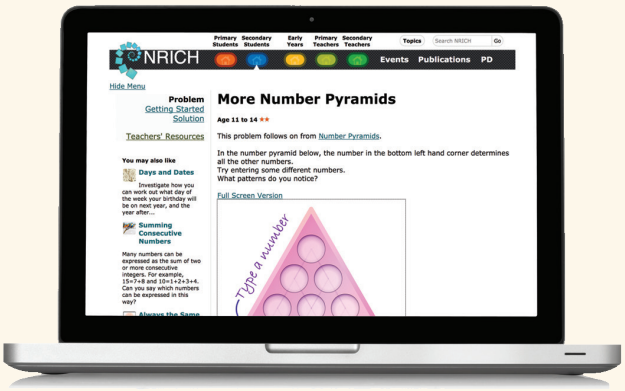
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students about the factorising, we can challenge students to transfer their knowledge about area to recognise the generalisation for themselves, by ***asking*** questions.

This digital object from the NRICH website generates triangular patterns, requiring students to identify the pattern by using different starting numbers. It also allows students to think more deeply about what numbers cannot be generated.

Students have the opportunity to apply their algebraic skills to better understand the patterns they observe. Challenges can be set and solved using algebraic equations.

The link to the problem on the NRICH website is: http://nrich.maths.org/2282



Example 7: Designing table mats and mosaics – distributive law

ACMNA190 ◆
Students extend and apply the distributive law to the expansion of algebraic expressions.

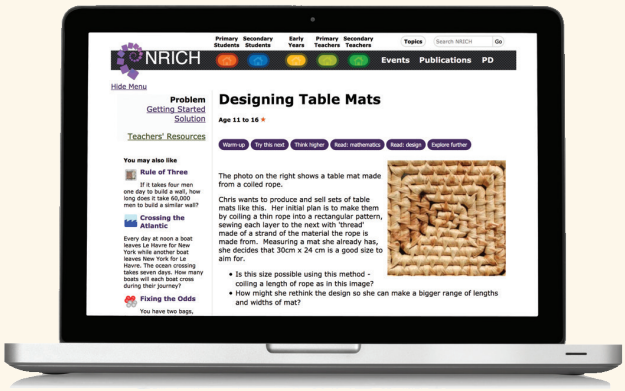
Questions from the BitL tool

Problem-solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.

Reasoning proficiency:
What can you infer?

Instead of ***telling*** students, we can challenge students to recognise the relationships between the events for themselves, by ***asking*** questions.

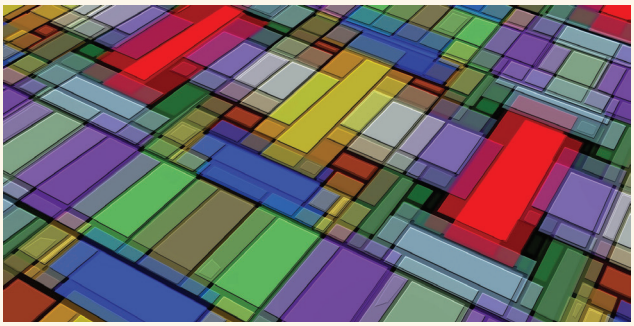
This open-ended task is from the NRICH website. It encourages students to develop their own mathematical model for designing rectangular table mats of given dimensions.



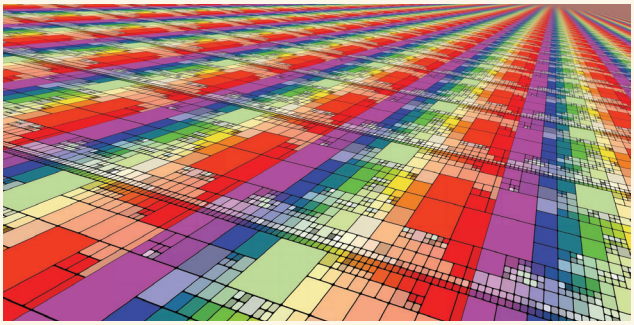
The link to the problem on the NRICH website is: https://nrich.maths.org/8252

Will they all fit together?

This activity could also be adapted as a class project, where students design the internal and overall dimensions of a tile mosaic made of individual units made from coiled rope. This could be planned in conjunction with another learning area.



Source: wobogre, Pixabay



Source: geralt, Pixabay

(A similar activity can be found in *Example 5: Factorising with tiles – area model.*)

Example 8: Stacking dice – finding a pattern

ACMNA190 ◆
Students extend and apply the distributive law to the expansion of algebraic expressions.

Questions from the BitL tool
Problem-solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students how to use linear relationships to solve problems, we can challenge them to identify the information for themselves, by ***asking*** questions.

Consider a stack of any number of dice with a '2' as the top face. Is there a way of knowing the sum of all the dice faces that you can not see?

Interpret

What are you trying to find out? What do you need to show to answer that question? What information is helpful? What information is not useful? What extra information do you want to collect? What information will you need/can you reasonably infer? (Remember that telling students or prompting heavily is stealing the opportunity for them to realise for themselves later on in the task. Not having a set number of dice is a stumbling block for some students. Ask what is making the problem difficult and what information would they like, then encourage them to decide the number of dice for themselves to get started. Setting the top number to always be 2, means there is only one variable determining the sum, which is 'n', the number of dice.)

Model and plan

Do you have an idea? How might you start? What equipment will be helpful? Would it help to start by thinking about a smaller version of this pattern? (Ask students to speak to someone who they think is being a good problem-solver today and ask them to show what they are trying. Students can identify from the faces they see which are missing and will total those, but this does not lead to generalisation. Putting totals for various different size stacks with a 2 on the top face into a table and/or graph, will help them recognise the number pattern emerging.)

Solve and check

Questions to be used only after students have grappled with the problem for a few minutes:

How might you answer the question for a small number of dice, say 4? Is there a more efficient way to find the sum? How would the sum change if you had 5 dice? What does this make you think? Does that seem right to you? Do other people think that too? (Observing the number patterns can reveal that adding another dice to the stack increases the sum by 7. It can be reasoned that this is because it also adds two unseen faces which are always opposite each other. A property of conventional dice is that the opposite faces sum to 7.)

Reflect

Pair up with someone who did it differently. How do your methods compare? What do you like about each other's strategy? How could you help each other to improve? Have you reached the same/a similar conclusion? How efficient was your strategy? Is there something that you would do differently next time? Is there a more efficient way? (Students may have written or verbal explanations, or a range of different algebraic expressions to determine the sum of the unseen faces, such as $7(n-1) + 5 = 7n-2$).

Check that these solutions are the same in as many ways as possible, such as using the distributive law, diagrammatic model, Excel spreadsheet, etc.

(This activity also appears in the *Linear and non-linear relationships: Year 8* narrative.)

Example 9: Cracking the concrete – finding a pattern

ACMNA192 ◆
Students simplify algebraic expressions involving the four operations.

Questions from the BitL tool
Problem-solving proficiency:
Interpret; Model and plan; Solve and check; Reflect.
Reasoning proficiency:
What can you infer?

Instead of ***telling*** students how to use linear relationships to solve problems, we can challenge them to identify the information for themselves, by ***asking*** questions.

An electrical contractor has to install a power point at position P, from the electrical supply at point O, using underground cabling across a pavement that is 4 pavers wide (see Figure 3). She will damage several concrete pavers in the process and these will need to be replaced. This cost must be included in her quote.

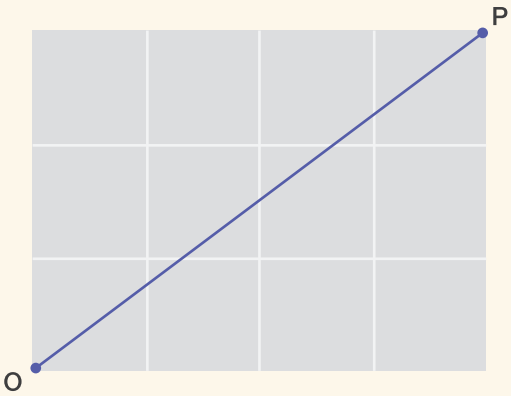


Figure 3

She finds this situation occurs a lot in her work and would like to have a way of determining how many pavers will be affected without counting them, particularly over long sections of pavement. Is that possible?

Interpret

What have you been asked to find out? What might you need to work out to answer that question? What information is helpful? What information is not useful? What extra information do you need? What information can you reasonably infer? (Remember that telling students or prompting heavily is stealing the opportunity for them to realise for themselves later on in the task.)

Model and plan

Do you have an idea? How might you start? Would it help if you thought about a similar problem for a simpler situation first? (Ask students to speak to someone who they think is being a good problem-solver today and ask them to show what they are trying.)

Solve and check

Are there some situations which could be considered similar to each other? What is the simplest type of situation? Does that seem right to you? Do other people think that too? What could you do to check your thinking? (For rectangles whose dimensions are relative prime, eg 4 by 3, the amount of broken pavers can be calculated as (4 + 3-1) = 6. In general, for relative prime dimensions **4** and **a**, the number of broken pavers is: **(4 + a-1) = a + 3**. This is because the line will crack **4** pavers from left to right. It will also crack **a - 1** pavers extra from bottom to top. For non-relative prime numbers, eg 6 by 8, this is calculated as 2 sets of 3 by 4, which will have damage ((4 + 3 – 1) x 2 = 12 pavers.)

Reflect

Pair up with someone who did it differently. How do your methods compare? What do you like about each other's strategy? How could you help each other to improve? Have you reached the same/a similar conclusion? How efficient was your strategy? Is there something that you would do differently next time? Is there a more efficient way? (How would your thinking change if the width of the path was different to 4? How would your thinking change if the paving pattern was like the one shown below?)

(This activity also appears in the *Linear and non-linear relationships: Year 8* narrative.)

Connections between ‘Patterns and algebra’ and other maths content

There are many opportunities to connect to other content in the AC: Mathematics, when we use patterns and algebra as a starting point.

Here are just some of the possible connections that can be made:

Mathematics: Year 8	
Whilst working with Patterns and algebra, connections can be made to:	**How the connection might be made:**
Students plot linear relationships on the Cartesian plane with and without the use of digital technologies. ACMNA193	Refer to: **Example 1: Hexagonal train – identifying equivalent algebraic expressions** **Example 8: Stacked dice – finding a pattern**
Students solve linear equations using algebraic and graphical techniques. Students verify solutions by substitution. ACMNA194	Refer to: **Example 1: Hexagonal train – identifying equivalent algebraic expressions** **Example 8: Stacked dice – finding a pattern**

Making connections to other learning areas

We know that when our students meet a concept frequently and in different contexts, they have a greater chance of developing understanding. With this in mind, it is our responsibility help our students to make these connections by intentionally designing tasks that connect a number of different content descriptions. Alternatively, connections can be made through questioning individual or small groups of students.

‘Patterns and algebra’ from Foundation to Year 10A

The AC: Mathematics year level content descriptions shown here have been colour coded to highlight the following curriculum aspects of working with Patterns and algebra:

Copy, continue and create patterns ◆
In **Foundation** through the ‘patterns and algebra content descriptions’, students copy, continue and create patterns.

Investigate and describe number patterns ◆
In **Years 1 to 6** students are expected to investigate and describe number patterns.

Use variables to represent numbers and create algebraic expressions ◆
In **Year 7** students mostly use variables to represent numbers and create algebraic expressions.

Simplify and identify equivalent algebraic expressions by extending and applying laws and properties of numbers ◆
In **Year 8 and Year 10** students mostly simplify and identify equivalent algebraic expressions by extending and applying laws and properties of numbers.

Use algebraic thinking and processes to solve problems ◆
In **Years 3 to 6**, students use algebraic thinking to solve problems. In **Year 10/10A** students mostly use algebraic thinking and processes to solve problems.

Year level	‘Patterns and algebra’ content descriptions from the AC: Mathematics
Foundation ◆	Students sort and classify familiar objects and explain the basis for these classifications. Students copy, continue and create patterns with objects and drawings. ACMNA005
Year 1 ◆	Students investigate and describe number patterns formed by skip-counting and patterns with objects. ACMNA018
Year 2 ◆ ◆	Students describe patterns with numbers and identify missing elements. ACMNA035
Year 2 ◆	Students solve problems by using number sentences for addition or subtraction. ACMNA036
Year 3 ◆	Students describe, continue, and create number patterns resulting from performing addition or subtraction. ACMNA060
Year 4 ◆	Students explore and describe number patterns resulting from performing multiplication. ACMNA081
Year 4 ◆	Students solve word problems by using number sentences involving multiplication or division where there is no remainder. ACMNA082
Year 4 ◆	Students find unknown quantities in number sentences involving addition and subtraction and identify equivalent number sentences involving addition and subtraction. ACMNA083
Year 5 ◆	Students describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction. ACMNA107
Year 5 ◆	Students find unknown quantities in number sentences involving multiplication and division and identify equivalent number sequences involving multiplication and division. ACMNA121
Year 6 ◆ ◆	Students continue and create sequences involving whole numbers, fractions and decimals. Students describe the rule used to create the sequence. ACMNA133
Year 6 ◆	Students explore the use of brackets and order of operations to write number sentences. ACMNA134
Year 7 ◆	Students introduce the concept of variables as a way of representing numbers using letters. ACMNA175

Year 7 ◆	Students create algebraic expressions and evaluate them by substituting a given value for each variable. ACMNA176
Year 7 ◆	Students extend and apply the laws and properties of arithmetic to algebraic terms and expressions. ACMNA177
Year 8 ◆	Students extend and apply the distributive law to the expansion of algebraic expressions. ACMNA190
Year 8 ◆	Students factorise algebraic expressions by identifying numerical factors. ACMNA191
Year 8 ◆	Students simplify algebraic expressions involving the four operations. ACMNA192
Year 9 ◆	Students extend and apply the index laws to variables, using positive integer indices and the zero index. ACMNA212
Year 9 ◆	Students apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate. ACMNA213
Year 10 ◆	Students factorise algebraic expressions by taking out a common algebraic factor. ACMNA230
Year 10 ◆	Students simplify algebraic products and quotients using index laws. ACMNA231
Year 10 ◆	Students apply the four operations to simple algebraic fractions with numerical denominators. ACMNA232
Year 10 ◆	Students expand binomial products and factorise monic quadratic expressions using a variety of strategies. ACMNA233
Year 10 ◆	Students substitute values into formulas to determine an unknown. ACMNA234
Year 10A ◆	Students investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems. ACMNA266

Numeracy continuum: Recognise and use patterns and relationships	
End Foundation	Describe and continue patterns.
End Year 2	Identify, describe and create everyday patterns.
End Year 4	Identify and describe trends in everyday patterns.
End Year 6	Identify and describe pattern rules and relationships that help to identify trends.
End Year 8	Identify trends using number rules and relationships.
End Year 10	Explain how the practical application of patterns can be used to identify trends.

Source: ACARA, Australian Curriculum: Mathematics

Resources

NRICH website

http://nrich.maths.org

In this conceptual narrative we have highlighted the possibility of using tasks from an organisation called NRICH enriching mathematics.

The NRICH website contains a large collection of high quality maths problem-solving tasks, together with suggestions about content that may be related to the task, ways to get started and different (valid) solutions that have been submitted by students from around the world.

Dan Meyer's blog: 101 questions

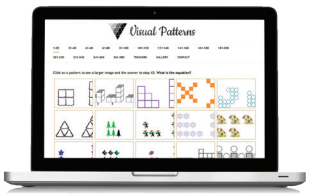
http://www.101qs.com

Dan's blog contains images and short films that can be presented to students along with the question: *What's the first question that comes to mind?*

A spreadsheet of ***Dan Meyer's Three-Act Maths Tasks*** can be accessed at http://bit.ly/DM3ActMathTasks.

Visual patterns

http://www.visualpatterns.org/



This website has multiple visual patterns that students can consider and describe in a way that makes sense to them. Students will have multiple interpretations of each image, promoting multiple ways of visualising, solving problems and stimulating dialogue. They can be used as lesson starters.

Scootle

https://www.scootle.edu.au/ec/p/home

This website has over 20,000 quality-assured digital learning resources aligned to the Australian Curriculum. You can filter your search to uncover a wealth of relevant teaching and learning items.

reSolve: maths by inquiry

https://www.resolve.edu.au

This website provides classroom resources for years F to 10 that promote fluency, deep understanding, strategic problem-solving, and mathematical reasoning. Each classroom resource is designed to develop progressive understanding through tasks that encourage a spirit of inquiry.

Plus Magazine

https://plus.maths.org

An online magazine which aims to introduce readers to the beauty and the practical applications of mathematics. It includes articles, teaching packages, puzzles and more. It has examples of mathematical modelling from scaffolded to open in authentic contexts.

Numeracy in the News

http://www.mercurynie.com.au/mathguys/mercury.htm

Numeracy in the News is a website containing 313 full-text newspaper articles from the Tasmanian paper, *The Mercury*. Other News Limited newspapers from around Australia are also available. The intention of the website is to raise the consciousness of students as critical readers of media reports, including statistical inference. The 'Teacher discussion' notes are a great example of how you can adapt student questions to suit articles from our local papers, such as *The Advertiser*.

TIMES modules

http://schools.amsi.org.au/times-modules/

TIMES modules are prepared by the Australian Mathematical Sciences Institute (AMSI) as part of The Improving Mathematics Education in Schools (TIMES) Project. The 'Data investigation and interpretation' module has been developed for Foundation to Year 10, and is a great knowledge source for teachers, modelling good analysis and inference reports.

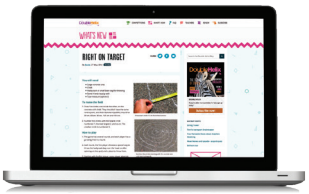
Top drawer teachers – resources for teachers of mathematics (statistics)

http://topdrawer.aamt.edu.au/Statistics

This website by the Australian Association of Mathematics Teachers, provides expert mathematics advice, teaching suggestions and classroom activities. Each 'drawer' is divided into sections: Big ideas, Misunderstandings, Good teaching, Assessment, and Activities.

Double Helix Extra

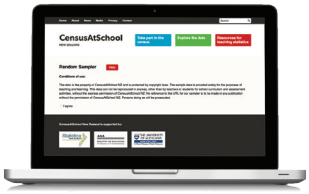
https://blog.doublehelix.csiro.au/



This CSIRO Double Helix Extra is a free fortnightly email newsletter featuring mathematics news and activities. It includes a quiz, brainteaser, news and a classic hands-on activity.

CensusAtSchool NZ

http://new.censusatschool.org.nz/tools/random-sampler/



CensusAtSchool NZ is part of a global project that provides a random sampler and additional resources for teaching statistics. It aims to:

- *'foster a positive attitude to statistics through using data that is both relevant and real*
- *improve understanding of a data gathering process, its purposes and benefits to society*
- *provide access to large and meaningful multivariate data sets*
- *encourage effective IT teaching and learning*
- *enhance the process of statistical enquiry across the curriculum.'*

Notes

Do you want to feel more confident about the maths you are teaching?

Do you want activities that support you to embed the proficiencies?

Do you want your students thinking mathematically rather than just doing maths?

If you answered **yes** to any of these questions, then this resource is for you.

Packed full of examples, along with questions you can ask students as they engage in their learning, this resource supports you to develop confidence in teaching the Australian Curriculum: Mathematics.

Produced by the Learning Improvement division

Excluded from NEALS